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FREE VIBRATION OF LAMINATED CONICAL SHELLS INCLUDING TRANSVERSE SHEAR DEFORMATION

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Abstract—The equilibrium equations for free vibration of composite laminated conical shells including the transverse shear deformation and the extension–bending coupling are formulated in terms of the five displacements using a particularly convenient coordinate system. The solutions for these five displacements are obtained, in the form of a power series, from the five governing equations, and the convergence condition for the solutions is also determined. Illustrative examples are presented to investigate the effect of the transverse shear deformation on the frequency parameters of axisymmetric vibration of isotropic and cross-plied laminated cones with different geometric and material parameters and under various types of boundary conditions.

1. INTRODUCTION

The free vibration of isotropic conical shells has been studied by many researchers using the Rayleigh–Ritz technique (Saunders *et al.*, 1960; Garnet and Kemper, 1964), the numerical integration method (Goldberg *et al.*, 1960; Kalnins, 1964; Irie *et al.*, 1982, 1984), the finite element method (Sen and Gould, 1974) and the analytical solution approach (Dreher and Leissa, 1970). In addition to the work done for free vibration of isotropic cones, there have been a few studies for free vibration of orthotropic conical shells (Cohen, 1965; Siu and Bert, 1970; Wilkins *et al.*, 1970; Yang, 1974) and composite laminated (Chandrasekaran and Ramamurti, 1982; Sankaranarayanan *et al.*, 1987, 1988; Tong and Wang, 1988; Tong, 1993). Tong (1993) obtained a simple and exact solution for the Donnell-type governing equations of free vibration of composite laminated conical shells including the extension–bending coupling terms.

In the present investigation, the solution procedure developed in Tong (1993) is utilized to study the free vibration of composite laminated conical shells including the effect of the transverse shear deformation, the rotatory inertia and the extension–bending coupling. For the classical shell theory, in which the transverse shear deformation is neglected, there are three displacement-type equilibrium equations governing the free vibration of composite laminated conical shells. For the improved shell theory, in which the transverse shear deformation is included, there are five displacement-type equilibrium equations in terms of five displacements for free vibration of composite laminated conical shells. Following the solution procedure developed for the classical shell theory (see Tong, 1993), a simple solution is obtained directly from the five governing equilibrium equations of displacement-type for free vibration of composite laminated conical shells including the transverse shear deformation, the rotatory inertia and the extension–bending coupling. Illustrative examples are given for the axisymmetric vibration of composite laminated cones to show the effect of the transverse shear deformation in terms of reducing the frequency parameters predicted by the classical shell theory.

2. GOVERNING EQUATIONS

Consider a composite laminated truncated circular conical shell, and let R_1 and R_2 indicate the radius of the cone at its small and large ends, respectively, α denotes semivertex angle of the cone and L is the cone length along its generator. We now introduce the x–ϕ coordinate system; x is measured along the cone's generator starting at middle length and

ϕ is the circumferential coordinate. The displacements of the shell's middle surface are denoted by U and V along x and ϕ directions, respectively, and by W along the normal to the surface (outward positive). The rotations of the normal are denoted by β_x and β_ϕ about ϕ and x axes, respectively. In terms of these variables the cone's radius at any point along its length may be expressed as:

$$R(x) = R_o + x \sin \alpha, \tag{1}$$

where R_o is the average radius of the cone.

For free vibration of composite laminated conical shells including the transverse shear deformation and the extension–bending coupling terms, the strain-displacement relations, the equilibrium equations and the stress resultant–strain equations are given as follows:

Strain-displacement relations:

$$\begin{aligned}
\varepsilon_x &= \frac{\partial U}{\partial x} \\
\varepsilon_\phi &= \frac{U \sin \alpha + W \cos \alpha}{R(x)} + \frac{1}{R(x)} \frac{\partial V}{\partial \phi} \\
\gamma_{x\phi} &= \frac{1}{R(x)} \frac{\partial U}{\partial \phi} - \frac{V \sin \alpha}{R(x)} + \frac{\partial V}{\partial x},
\end{aligned} \tag{2a}$$

$$\begin{aligned}
\kappa_x &= \frac{\partial \beta_x}{\partial x} \\
\kappa_\phi &= \frac{1}{R(x)} \left(\frac{\partial \beta_\phi}{\partial \phi} + \beta_x \sin \alpha \right) \\
\kappa_{x\phi} &= \frac{\partial \beta_\phi}{\partial x} - \frac{\beta_\phi \sin \alpha}{R(x)} + \frac{1}{R(x)} \frac{\partial \beta_x}{\partial \phi},
\end{aligned} \tag{2b}$$

$$\begin{aligned}
\varepsilon_{xz} &= \beta_x + \frac{\partial W}{\partial x} \\
\varepsilon_{\phi z} &= \beta_\phi - \frac{V \cos \alpha}{R(x)} + \frac{1}{R(x)} \frac{\partial W}{\partial \phi}.
\end{aligned} \tag{2c}$$

Equilibrium equations:

$$\begin{aligned}
\frac{\partial N_x}{\partial x} + \frac{\sin \alpha}{R(x)} (N_x - N_\phi) + \frac{1}{R(x)} \frac{\partial N_{x\phi}}{\partial \phi} &= \rho h \ddot{U} \\
\frac{\partial N_{x\phi}}{\partial x} + \frac{2 \sin \alpha}{R(x)} N_{x\phi} + \frac{1}{R(x)} \frac{\partial N_\phi}{\partial \phi} + \frac{\cos \alpha}{R(x)} Q_\phi &= \rho h \ddot{V} \\
\frac{\partial Q_x}{\partial x} + \frac{1}{R(x)} \frac{\partial Q_\phi}{\partial \phi} + \frac{\sin \alpha}{R(x)} Q_x - \frac{\cos \alpha}{R(x)} N_\phi &= \rho h \ddot{W} \\
\frac{\partial M_x}{\partial x} + \frac{\sin \alpha}{R(x)} (M_x - M_\phi) + \frac{1}{R(x)} \frac{\partial M_{x\phi}}{\partial \phi} - Q_x &= \frac{\rho h^3}{12} \ddot{\beta}_x \\
\frac{\partial M_{x\phi}}{\partial x} + \frac{2 \sin \alpha}{R(x)} M_{x\phi} + \frac{1}{R(x)} \frac{\partial M_\phi}{\partial \phi} - Q_\phi &= \frac{\rho h^3}{12} \ddot{\beta}_\phi,
\end{aligned} \tag{3}$$

where:

$$\rho = \frac{1}{h}\sum_{k=1}^{N_L} \rho_k(h_k - h_{k-1}). \tag{4}$$

Stress resultant–strain relations: it is assumed that there are no stretching–shearing, twisting–shearing, bending–shearing, and bending–twisting couplings, thus:

$$\begin{aligned}
N_x &= A_{11}\varepsilon_x + A_{12}\varepsilon_\phi + B_{11}\kappa_x + B_{12}\kappa_\phi \\
N_\phi &= A_{12}\varepsilon_x + A_{22}\varepsilon_\phi + B_{12}\kappa_x + B_{22}\kappa_\phi \\
N_{x\phi} &= A_{33}\gamma_{x\phi} + B_{33}\kappa_{x\phi},
\end{aligned} \tag{5a}$$

$$\begin{aligned}
M_x &= B_{11}\varepsilon_x + B_{12}\varepsilon_\phi + D_{11}\kappa_x + D_{12}\kappa_\phi \\
M_\phi &= B_{12}\varepsilon_x + B_{22}\varepsilon_\phi + D_{12}\kappa_x + D_{22}\kappa_\phi \\
M_{x\phi} &= B_{33}\gamma_{x\phi} + D_{33}\kappa_{x\phi},
\end{aligned} \tag{5b}$$

$$\begin{aligned}
Q_x &= A_{55}\varepsilon_{xz} \\
Q_\phi &= A_{44}\varepsilon_{\phi z},
\end{aligned} \tag{5c}$$

where:

$$A_{ij} = \sum_{k=1}^{N_L} \bar{Q}_{ij}^{(k)}(h_k - h_{k-1}) \qquad B_{ij} = \frac{1}{2}\sum_{k=1}^{N_L} \bar{Q}_{ij}^{(k)}(h_k^2 - h_{k-1}^2)$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N_L} \bar{Q}_{ij}^{(k)}(h_k^3 - h_{k-1}^3) \qquad i,j = 1,2,3, \tag{6a}$$

$$(A_{44}, A_{55}) = \sum_{k=1}^{N_L} \int_{h_{k-1}}^{h_k} (\bar{Q}_{44}^{(k)}, \bar{Q}_{55}^{(k)}) f(z)\,\mathrm{d}z, \tag{6b}$$

and

$$f(z) = \frac{5}{4}\left[1 - 4\left(\frac{z}{h}\right)^2\right], \tag{7}$$

where h is the total wall thickness of the shell.

On substituting eqns (2) into eqns (5), and further into eqns (3), we may express the governing equations for free vibration analysis of laminated conical shells in terms of the five displacements, namely:

$$\begin{aligned}
L_{11}U + L_{12}V + L_{13}W + L_{14}\beta_x + L_{15}\beta_\phi &= \rho h\ddot{U} \\
L_{21}U + L_{22}V + L_{23}W + L_{24}\beta_x + L_{25}\beta_\phi &= \rho h\ddot{V} \\
L_{31}U + L_{32}V + L_{33}W + L_{34}\beta_x + L_{35}\beta_\phi &= \rho h\ddot{W} \\
L_{41}U + L_{42}V + L_{43}W + L_{44}\beta_x + L_{45}\beta_\phi &= \frac{\rho h^3}{12}\ddot{\beta}_x \\
L_{51}U + L_{52}V + L_{53}W + L_{54}\beta_x + L_{55}\beta_\phi &= \frac{\rho h^3}{12}\ddot{\beta}_\phi,
\end{aligned} \tag{8}$$

where

$$L_{11} = A_{11}\frac{\partial^2}{\partial x^2} + \frac{A_{11}\sin\alpha}{R(x)}\frac{\partial}{\partial x} - \frac{A_{22}\sin^2\alpha}{R^2(x)} + \frac{A_{33}}{R^2(x)}\frac{\partial^2}{\partial \phi^2}$$

$$L_{12} = \frac{(A_{12}+A_{33})}{R(x)}\frac{\partial^2}{\partial x\,\partial\phi} - \frac{(A_{22}+A_{33})\sin\alpha}{R^2(x)}\frac{\partial}{\partial\phi}$$

$$L_{13} = \frac{A_{12}\cos\alpha}{R(x)}\frac{\partial}{\partial x} - \frac{A_{22}\sin\alpha\cos\alpha}{R^2(x)}$$

$$L_{14} = B_{11}\frac{\partial^2}{\partial x^2} + \frac{B_{11}\sin\alpha}{R(x)}\frac{\partial}{\partial x} - \frac{B_{22}\sin^2\alpha}{R^2(x)} + \frac{B_{33}}{R^2(x)}\frac{\partial^2}{\partial\phi^2}$$

$$L_{15} = \frac{(B_{12}+B_{33})}{R(x)}\frac{\partial^2}{\partial x\,\partial\phi} - \frac{(B_{22}+B_{33})\sin\alpha}{R^2(x)}\frac{\partial}{\partial\phi}$$

$$L_{21} = \frac{(A_{12}+A_{33})}{R(x)}\frac{\partial^2}{\partial x\,\partial\phi} + \frac{(A_{22}+A_{33})\sin\alpha}{R^2(x)}\frac{\partial}{\partial\phi}$$

$$L_{22} = A_{33}\left[\frac{\partial^2}{\partial x^2} + \frac{\sin\alpha}{R(x)}\frac{\partial}{\partial x} - \frac{\sin^2\alpha}{R^2(x)}\right] + \frac{A_{22}}{R^2(x)}\frac{\partial^2}{\partial\phi^2} - \frac{A_{44}\cos^2\alpha}{R^2(x)}$$

$$L_{23} = \frac{(A_{22}+A_{44})\cos\alpha}{R^2(x)}\frac{\partial}{\partial\phi}$$

$$L_{24} = \frac{(B_{12}+B_{33})}{R(x)}\frac{\partial^2}{\partial x\,\partial\phi} + \frac{(B_{22}+B_{33})\sin\alpha}{R^2(x)}\frac{\partial}{\partial\phi}$$

$$L_{25} = B_{33}\left[\frac{\partial^2}{\partial x^2} + \frac{\sin\alpha}{R(x)}\frac{\partial}{\partial x} - \frac{\sin^2\alpha}{R^2(x)}\right] + \frac{B_{22}}{R^2(x)}\frac{\partial^2}{\partial\phi^2} + \frac{A_{44}\cos\alpha}{R(x)}$$

$$L_{31} = -\frac{A_{12}\cos\alpha}{R(x)}\frac{\partial}{\partial x} - \frac{A_{22}\sin\alpha\cos\alpha}{R^2(x)}$$

$$L_{32} = -L_{23}$$

$$L_{33} = A_{55}\left[\frac{\partial^2}{\partial x^2} + \frac{\sin\alpha}{R(x)}\frac{\partial}{\partial x}\right] + \frac{A_{44}}{R^2(x)}\frac{\partial^2}{\partial\phi^2} - \frac{A_{22}\cos^2\alpha}{R^2(x)}$$

$$L_{34} = \left(A_{55} - \frac{B_{12}\cos\alpha}{R(x)}\right)\frac{\partial}{\partial x} + \frac{A_{55}\sin\alpha}{R(x)} - \frac{B_{22}\sin\alpha\cos\alpha}{R^2(x)}$$

$$L_{35} = \left(\frac{A_{44}}{R(x)} - \frac{B_{22}\cos\alpha}{R^2(x)}\right)\frac{\partial}{\partial\phi}$$

$$L_{41} = L_{14}$$

$$L_{42} = \frac{(B_{12}+B_{33})}{R(x)}\frac{\partial^2}{\partial x\,\partial\phi} - \frac{(B_{22}+B_{33})\sin\alpha}{R^2(x)}\frac{\partial}{\partial\phi}$$

$$L_{43} = -\left(A_{55} - \frac{B_{12}\cos\alpha}{R(x)}\right)\frac{\partial}{\partial x} - \frac{B_{22}\sin\alpha\cos\alpha}{R^2(x)}$$

$$L_{44} = D_{11}\frac{\partial^2}{\partial x^2} + \frac{D_{11}\sin\alpha}{R(x)}\frac{\partial}{\partial x} - \frac{D_{22}\sin^2\alpha}{R^2(x)} + \frac{D_{33}}{R^2(x)}\frac{\partial^2}{\partial\phi^2} - A_{55}$$

$$L_{45} = \frac{(D_{12}+D_{33})}{R(x)}\frac{\partial^2}{\partial x\,\partial\phi} - \frac{(D_{22}+D_{33})\sin\alpha}{R^2(x)}\frac{\partial}{\partial\phi}$$

$$L_{51} = \frac{(B_{12}+B_{33})}{R(x)}\frac{\partial^2}{\partial x\,\partial\phi} + \frac{(B_{22}+B_{33})\sin\alpha}{R^2(x)}\frac{\partial}{\partial\phi}$$

$$L_{52} = L_{25} \quad L_{53} = -L_{35}$$

$$L_{54} = \frac{(D_{12}+D_{33})}{R(x)}\frac{\partial^2}{\partial x\,\partial\phi} + \frac{(D_{22}+D_{33})\sin\alpha}{R^2(x)}\frac{\partial}{\partial\phi}$$

$$L_{55} = D_{33}\left[\frac{\partial^2}{\partial x^2} + \frac{\sin\alpha}{R(x)}\frac{\partial}{\partial x} - \frac{\sin^2\alpha}{R^2(x)}\right] + \frac{D_{22}}{R^2(x)}\frac{\partial^2}{\partial\phi^2} - A_{44}. \tag{9}$$

The above set of governing equations degenerate to those of cylindrical shells when α is set equal to zero. It is also worth noting that if the starting point of the x axis is changed to the cone's vertex, where the radius is equal to zero, the x–ϕ coordinate system will coincide with the s–ϕ coordinate system used by many previous researchers.

The related boundary conditions at both ends of the cone may be expressed generally as:

$$\begin{aligned} N_x &= 0 \quad \text{or} \quad U = 0 \\ N_{x\phi} &= 0 \quad \text{or} \quad V = 0 \\ Q_x &= 0 \quad \text{or} \quad W = 0 \\ M_x &= 0 \quad \text{or} \quad B_x = 0 \\ M_{x\phi} &= 0 \quad \text{or} \quad \beta_\phi = 0. \end{aligned} \tag{10}$$

Evidently the governing equations presented in the foregoing are complex and to our knowledge exact solutions have not been given for these equations. In the following section we outline a strategy for constructing general solutions for these equations.

3. SOLUTIONS

Following the solution procedure outlined in Tong (1993), let us assume solutions for eqns (8), of the following form:

$$\begin{aligned} U &= \sum_{m=0}^{\infty} a_m x^m \cos n\phi\, e^{i\omega t} \qquad V = \sum_{m=0}^{\infty} b_m x^m \sin n\phi\, e^{i\omega t} \\ W &= \sum_{m=0}^{\infty} c_m x^m \cos n\phi\, e^{i\omega t} \\ \beta_x &= \sum_{m=0}^{\infty} d_m x^m \cos n\phi\, e^{i\omega t} \qquad \beta_\phi = \sum_{m=0}^{\infty} e_m x^m \sin n\phi\, e^{i\omega t}. \end{aligned} \tag{11}$$

Where $i = \sqrt{-1}$ and n is an integer representing the circumferential wave number of the conical shell, a_m, b_m, c_m, d_m and e_m are constants to be determined later.

On substituting eqns (11) into eqns (8), which are modified by multiplying all the equations with $R^2(x)$, and using eqns (1) and (9), five linear algebraic equations, developed by matching the terms of same order in x, are obtained and further rewritten as the following recurrence relations:

$$\begin{aligned} a_{m+2} = {} & F_{1,1}a_{m+1} + F_{1,2}a_m + F_{1,3}a_{m-1} + F_{1,4}a_{m-2} + F_{1,5}b_{m+1} + F_{1,6}b_m \\ & + F_{1,7}c_{m+1} + F_{1,8}c_m + F_{1,9}c_{m-1} + F_{1,10}d_{m+1} + F_{1,11}d_m + F_{1,12}d_{m-1} \\ & + F_{1,13}d_{m-2} + F_{1,14}e_{m+1} + F_{1,15}e_m, \end{aligned} \tag{12a}$$

$$\begin{aligned} b_{m+2} = {} & F_{2,1}a_{m+1} + F_{2,2}a_m + F_{2,3}b_{m+1} + F_{2,4}b_m + F_{2,5}b_{m-1} + F_{2,6}b_{m-2} \\ & + F_{2,7}c_m + F_{2,8}c_{m-1} + F_{2,9}d_{m+1} + F_{2,10}d_m + F_{2,11}e_{m+1} + F_{2,12}e_m \\ & + F_{2,13}e_{m-1} + F_{2,14}e_{m-2}, \end{aligned} \tag{12b}$$

$$\begin{aligned} c_{m+2} = {} & F_{3,1}a_{m+1} + F_{3,2}a_m + F_{3,3}b_m + F_{3,4}c_{m+1} + F_{3,5}c_m + F_{3,6}c_{m-1} \\ & + F_{3,7}c_{m-2} + F_{3,8}d_{m+1} + F_{3,9}d_m + F_{3,10}d_{m-1} + F_{3,11}e_m + F_{3,12}e_{m-1}, \end{aligned} \tag{12c}$$

$$d_{m+2} = F_{4,1}a_{m+1} + F_{4,2}a_m + F_{4,3}a_{m-1} + F_{4,4}a_{m-2} + F_{4,5}b_{m+1} + F_{4,6}b_m + F_{4,7}c_{m+1} + F_{4,8}c_m + F_{4,9}c_{m-1} + F_{4,10}d_{m+1} + F_{4,11}d_m + F_{4,12}d_{m-1} + F_{4,13}d_{m-2} + F_{4,14}e_{m+1} + F_{4,15}e_m, \quad (12d)$$

$$e_{m+2} = F_{5,1}a_{m+1} + F_{5,2}a_m + F_{5,3}b_{m+1} + F_{5,4}b_m + F_{5,5}b_{m-1} + F_{5,6}b_{m-2} + F_{5,7}c_m + F_{5,8}c_{m-1} + F_{5,9}d_{m+1} + F_{5,10}d_m + F_{5,11}e_{m+1} + F_{5,12}e_m + F_{5,13}e_{m-1} + F_{5,14}e_{m-2}, \quad (12e)$$

where the coefficients $F_{i,j}$ ($(i, j) = (1, 15)$, $(2, 14)$, $(3, 12)$, $(4, 15)$ and $(5, 14)$) are given in the Appendix. The above recurrence relations allow us to express the unknown constants in terms of a_k, b_k, c_k, d_k, and e_k ($k = 0, 1$), which are the unknowns to be determined by imposing the boundary conditions at both ends of the cone.

Before going into details of the solution procedure, let us consider the convergence condition of the series solutions defined in eqns (11) and the associated recurrence eqns (12).

Careful analysis of eqns (12) and the coefficients $F_{i,j}$ given in the Appendix shows that: (a) the power series defined in eqns (11) are alternating series, i.e. the terms of the series change sign consecutively; (b) when m is large enough, eqns (12) can be written approximately as follows:

$$A_{11}\left(a_{m+2} + \frac{2\sin\alpha}{R_o}a_{m+1} + \frac{\sin^2\alpha}{R_o^2}a_m\right) + B_{11}\left(d_{m+2} + \frac{2\sin\alpha}{R_o}d_{m+1} + \frac{\sin^2\alpha}{R_o^2}d_m\right) = 0$$

$$A_{33}\left(b_{m+2} + \frac{2\sin\alpha}{R_o}b_{m+1} + \frac{\sin^2\alpha}{R_o^2}b_m\right) + B_{33}\left(e_{m+2} + \frac{2\sin\alpha}{R_o}e_{m+1} + \frac{\sin^2\alpha}{R_o^2}e_m\right) = 0$$

$$c_{m+2} + \frac{2\sin\alpha}{R_o}c_{m+1} + \frac{\sin^2\alpha}{R_o^2}c_m = 0$$

$$B_{11}\left(a_{m+2} + \frac{2\sin\alpha}{R_o}a_{m+1} + \frac{\sin^2\alpha}{R_o^2}a_m\right) + D_{11}\left(d_{m+2} + \frac{2\sin\alpha}{R_o}d_{m+1} + \frac{\sin^2\alpha}{R_o^2}d_m\right) = 0$$

$$B_{33}\left(b_{m+2} + \frac{2\sin\alpha}{R_o}b_{m+1} + \frac{\sin^2\alpha}{R_o^2}b_m\right) + D_{33}\left(e_{m+2} + \frac{2\sin\alpha}{R_o}e_{m+1} + \frac{\sin^2\alpha}{R_o^2}e_m\right) = 0. \quad (13)$$

These equations may be simplified as:

$$S_{m+2} + \frac{2\sin\alpha}{R_o}S_{m+1} + \frac{\sin^2\alpha}{R_o^2}S_m = 0 \quad (S = a, b, c, d, e). \quad (14)$$

These approximate equations indicate that the coefficients a_m, b_m, c_m, d_m, and e_m, are predominantly dependent on the former terms expressed in terms of a_k, b_k, c_k, d_k, and e_k, ($k = 0, 1$), respectively, when m is large enough.

Assuming the convergence ratio of U, V, W, β_x and β_ϕ to be ρ_a, ρ_b, ρ_c, ρ_d and ρ_e, respectively, i.e.

$$\rho_S = \lim_{m\to\infty}\frac{S_{m+1}}{S_m} \quad (S = a, b, c, d, e), \quad (15)$$

and noting the elementary character of the alternate series, eqn (14) can be changed into the following form:

$$\rho_S^2 = \frac{2\sin\alpha}{R_o}\rho_S + \frac{\sin^2\alpha}{R_o^2}\rho_s \quad (S = a, b, c, d, e), \tag{16}$$

which yields the following identical real root:

$$\rho_S = \frac{\sin\alpha}{R_o} \quad (S = a, b, c, d, e). \tag{17}$$

Therefore, the solutions for U, V, W, β_x and β_ϕ in eqns (11) have identical convergence radii. That is, as long as x is within the circle of the convergence radius, convergence of these five series will be assured. For the shells considered here, noting that the maximum value of x is $L/2$ and R_o is the average radius, for our purposes the condition for convergence can be finally rewritten as:

$$R_1 \geqslant 0. \tag{18}$$

Hence the five constructed series will converge to their corresponding solutions if the small radius of the cone is not zero, i.e., if the conical shell is a truncated one. A complete cone may be treated as a truncated cone with a very small radius at its apex. Thus for all practical purposes, there are no limitations on the geometric parameters of the shell considered. Accordingly, the solutions obtained provide exact solutions for the five displacements U, V, W, β_x and β_ϕ for the free vibrations of cones, and these five displacements may be used to calculate the stress resultants N_x, N_ϕ and $N_{x\phi}$, the bending moments M_x, M_ϕ and $M_{x\phi}$ and the transverse shear forces Q_x and Q_ϕ through eqns (2) and (5). This solution is exact because it satisfies the governing equations rigorously and it also satisfies the 10 boundary conditions through 10 arbitrary constants.

The free vibration frequencies and the related vibration modes can finally be obtained by equating the determinants of the coefficients matrix obtained after imposition of the 10 boundary conditions to zero.

4. ILLUSTRATIVE EXAMPLE

In this section, the foregoing theory is illustrated through numerical investigation of axisymmetric vibration of isotropic and laminated cones. The transverse shear deformation effect on the axisymmetric frequency parameters of the cones with various semivertex angles α, the slant lengths L and material properties is highlighted through comparing with those predicted by the classical shell theory (see Tong, 1993). In all calculations, 30 terms in all series were used. Before presenting the results, let us introduce the following notations:

$$\omega_{pi} = \sqrt{\frac{\rho h}{A_{11}}}\omega_i R_2 \quad \omega_{pc} = \sqrt{\frac{\rho h}{A_{11}}}\omega_c R_2. \tag{19}$$

Where ω_c and ω_{pc} are referred to as the frequency and its parameter of the cone computed using the classical shell theory (see Tong, 1993), ω_i and ω_{pi} are those calculated using the improved shell theory.

There are four types of boundary conditions used in the present study, and they are listed in Table 1.

Table 1. Boundary conditions

Type number	Improved shell theory including transverse shear deformation	Classical shell theory neglecting transverse shear deformation
1	$U = W = \beta_2 = V = M_{x\phi} = 0$ at both ends	$U = W = W_x = V = 0$ at both ends
2	$N_x = W = M_x = V = M_{x\phi} = 0$ at small end	$N_x = W = M_x = V = 0$ at small end
	$U = W = \beta_x = V = M_{x\phi} = 0$ at large end	$U = W = W_x = V = 0$ at large end
3	$U = W = \beta_x = V = M_{x\phi} = 0$ at small end	$U = W = W_x = V = 0$ at small end
	$N_x = W = M_x = V = M_{x\phi} = 0$ at large end	$N_x = W = M_x = V = 0$ at large end
4	$N_x = W = M_x = V = M_{x\phi} = 0$ at both ends	$N_x = W = M_x = V = 0$ at both ends

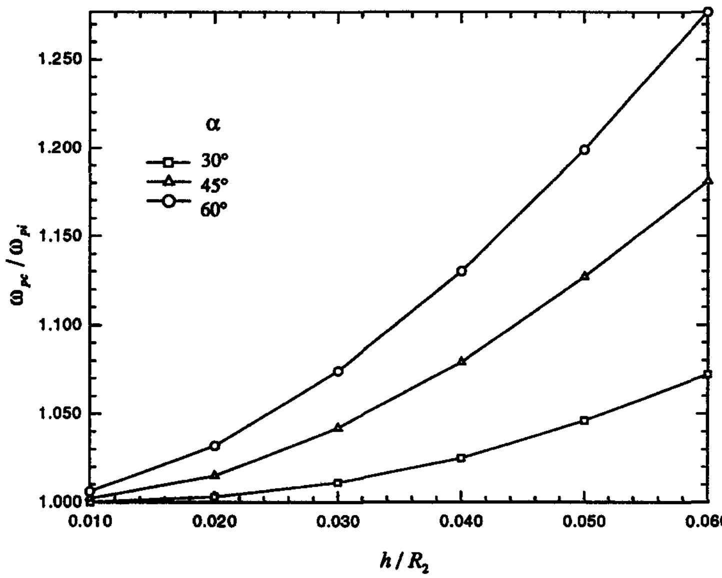

Fig. 1. ω_{pi}/ω_{pc} vs h/R_2 for isotropic cones with boundary condition of type 1 ($L \sin \alpha/R_2 = 0.25$).

The first example is analysis of axisymmetric vibration of isotropic cones with $\mu = 0.3$. The ratio ω_{pc}/ω_{pi} is plotted vs h/R_2 in Figs 1–4 for cones with different α, $L \sin \alpha/R_2$ and various types of boundary conditions; namely, Figs 1 and 2 with boundary condition type 1, Fig. 3 with type 2 and Fig. 4 with type 3. It can be seen that ω_{pc}/ω_{pi} increases as h/R_2 becomes large, namely, for thick shells, the results given by improved theory are smaller than those predicted by classical theory, while for thin shells both theories give almost the same results. For example, ω_{pi} is 1% larger than ω_{pc} for $h/R_2 = 0.01$, and 50% larger than ω_{pc} for $h/R_2 = 0.09$ for cones with $\alpha = 60°$, $L \sin \alpha/R_2 = 0.25$ and boundary condition type 1 (see Fig. 2). It is worth noting that cones with boundary condition type 1 tend to yield a larger percentage of reduction in the frequency parameter than those with type 2 and 3. Another phenomena worth noting is that the slant length of the cone decreases the ratio ω_{pc}/ω_{pi} when all other parameters and conditions remain unchanged, namely, long shell tends to lessen the transverse shear deformation effect in terms of reducing the frequency parameters computed using the classical shell theory.

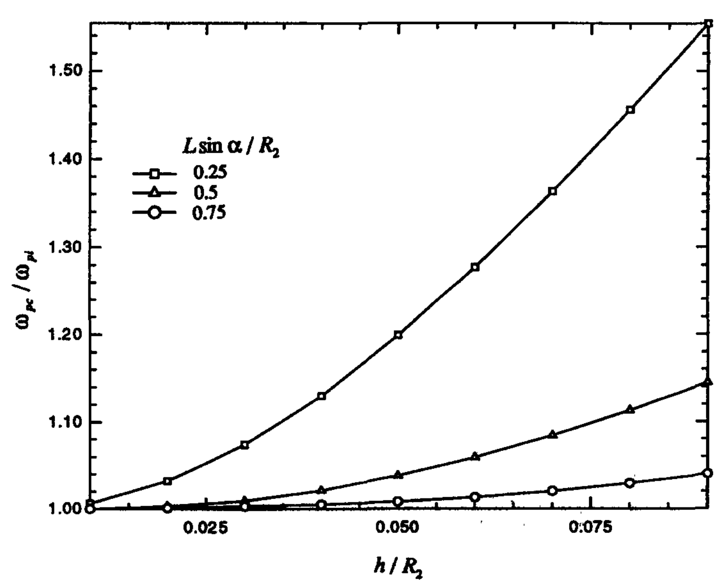

Fig. 2. ω_{pi}/ω_{pc} vs h/R_2 for isotropic cones with boundary condition of type 1 ($\alpha = 60°$).

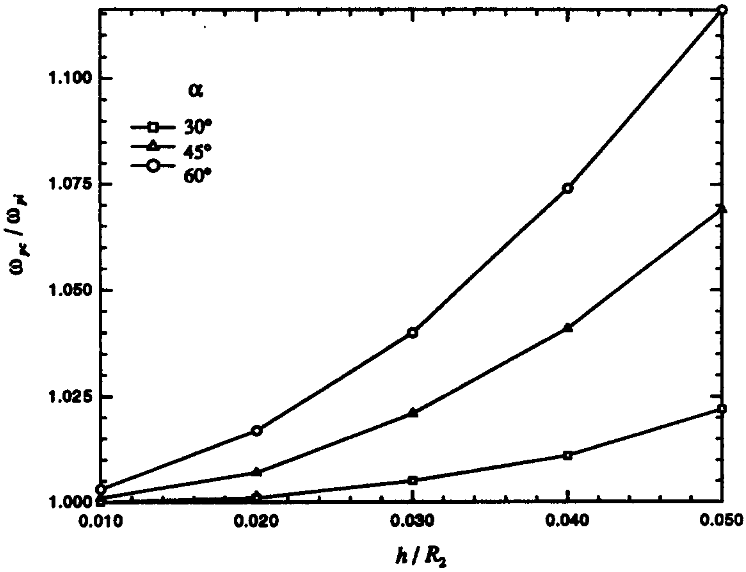

Fig. 3. ω_{pl}/ω_{pc} vs h/R_2 for isotropic cones with boundary condition of type 2 ($L \sin\alpha/R_2 = 0.25$).

The second illustrative example considers the axisymmetric vibration of antisymmetric cross-plied laminated cones with material parameters of each layer given as:

$$\frac{E_x}{E_\phi} = 15.0 \qquad \mu_{x\phi} = 0.25 \qquad \frac{G_{x\phi}}{E_\phi} = 0.5$$

$$\mu_{xz} = \mu_{\phi z} = 0.3 \quad G_{xz} = \frac{E_\phi}{2(1+\mu_{xz})} \quad G_{\phi z} = \frac{E_\phi}{2(1+\mu_{\phi z})}. \tag{20}$$

Thus the coefficients in eqns (5) are:

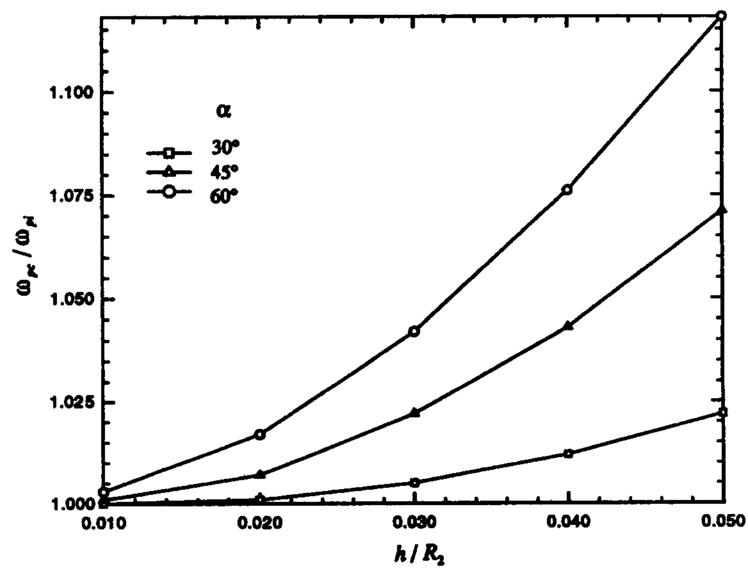

Fig. 4. ω_{pl}/ω_{pc} vs h/R_2 for isotropic cones with boundary condition of type 3 ($L \sin\alpha/R_2 = 0.25$).

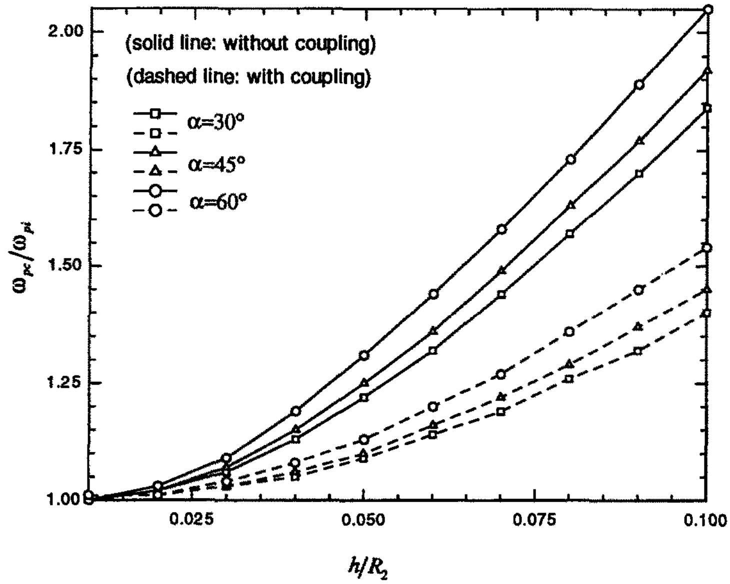

Fig. 5. ω_{pc}/ω_{pi} vs h/R_2 for laminated cones with boundary condition of type 1 ($L/R_2 = 0.25$).

$$
\begin{aligned}
&A_{11} = A_{22} = \tfrac{1}{2}(Q_{11}+Q_{22})h \qquad &&A_{12} = Q_{12}h \qquad &&A_{66} = Q_{66}h \\
&B_{11} = -B_{22} = \frac{1}{4\bar{N}}(Q_{11}-Q_{22})h^2 \qquad &&B_{12} = B_{66} = 0 && \\
&D_{11} = D_{22} = \tfrac{1}{24}(Q_{11}+Q_{22})h^3 \qquad &&D_{12} = \tfrac{1}{12}Q_{12}h^3 \qquad &&D_{66} = \tfrac{1}{12}Q_{66}h^3 \\
&A_{44} = \tfrac{5}{6}Q_{44}h &&A_{55} = \tfrac{5}{6}Q_{55}h, &&
\end{aligned} \tag{21}
$$

where h denotes the total thickness of the cone, and

$$
\begin{aligned}
&Q_{11} = \frac{E_x}{1-\mu_{x\phi}\mu_{\phi x}} \quad &&Q_{12} = \frac{\mu_{x\phi}E_\phi}{1-\mu_{x\phi}\mu_{\phi x}} \quad &&Q_{22} = \frac{E_\phi}{1-\mu_{x\phi}\mu_{\phi x}} \\
&Q_{44} = G_{\phi z} &&Q_{55} = G_{xz} &&Q_{66} = G_{x\phi}.
\end{aligned} \tag{22}
$$

It is worth pointing out that the extension–bending coupling terms attain their maximum value with two plies and become zero with an infinite number of plies. In this study, two cases are considered, namely, with coupling and without coupling. For the case of with coupling, we have $B_{11} = -B_{22} = (Q_{11}-Q_{22})h^2/8$, while for the other case, $B_{11} = B_{22} = 0$. When the total thickness of the shell is increased to study the transverse shear deformation effect, we assume that increase of the total thickness is realized through increasing the thickness of each layer of the two plies for the case of with coupling, and through increasing the total number of plies for the other case.

All calculations in the second example show that the extension–bending coupling decreases the frequency parameter ω_{pc} for axisymmetric vibration of the composite cones. This phenomenon is the same as that observed when using classical shell theory (see Tong, 1993). To show the effect of the transverse shear deformation, the ratio ω_{pc}/ω_{pi} is plotted vs h/R_2 in Fig. 5 for axisymmetric vibration of antisymmetric cross-plied laminated cones with $L/R_2 = 0.5$, different α and boundary condition of type 1. In this figure, the dashed and solid curves represent the results with and without coupling, respectively. It is observed that the dashed curves remain below the solid curves, and ω_{pc}/ω_{pi} increases as α becomes large. When $h/R_2 = 0.01$, ω_{pc}/ω_{pi} is almost equal to unity, which means that results from both theories are identical; when $h/R_2 = 0.1$, ω_{pc}/ω_{pi} is larger than 1.0, which shows that

Table 2. Frequency parameters ω_p for cones with SS3 ($\alpha = 30°$, $L/R_2 = 0.5$)

h/R_2	Without coupling ω_{pi}	ω_{pc}	ω_{pc}/ω_{pi}	With coupling ω_{pi}	ω_{pc}	ω_{pc}/ω_{pi}
0.01	0.1959	0.1978	1.010	0.1768	0.1769	1.000
0.02	0.2318	0.2355	1.016	0.2091	0.2119	1.013
0.03	0.2608	0.2671	1.024	0.2304	0.2360	1.024
0.04	0.2884	0.2992	1.037	0.2495	0.2578	1.033
0.05	0.3137	0.3308	1.054	0.2681	0.2794	1.042
0.06	0.3358	0.3606	1.074	0.2862	0.3010	1.052
0.07	0.3547	0.3877	1.093	0.3033	0.3222	1.062
0.08	0.3704	0.4117	1.111	0.3193	0.3426	1.073
0.09	0.3835	0.4325	1.128	0.3338	0.3620	1.084
0.10	0.3943	0.4504	1.142	0.3469	0.3801	1.096

Table 3. Frequency parameters ω_p for cones with SS3 ($\alpha = 45°$, $L/R_2 = 0.5$)

h/R_2	Without coupling ω_{pi}	ω_{pc}	ω_{pc}/ω_{pi}	With coupling ω_{pi}	ω_{pc}	ω_{pc}/ω_{pi}
0.01	0.2535	0.2556	1.008	0.2270	0.2321	1.022
0.02	0.3041	0.3084	1.014	0.2692	0.2797	1.039
0.03	0.3504	0.3590	1.025	0.3000	0.3164	1.055
0.04	0.3965	0.4131	1.042	0.3302	0.3529	1.069
0.05	0.4396	0.4679	1.062	0.3610	0.3905	1.082
0.06	0.4779	0.5207	1.089	0.3914	0.4290	1.096
0.07	0.5109	0.5695	1.115	0.4205	0.4673	1.111
0.08	0.5388	0.6129	1.137	0.4478	0.5046	1.127
0.09	0.5620	0.6505	1.157	0.4728	0.5403	1.143
0.10	0.5813	0.6823	1.174	0.4955	0.5738	1.158

Table 4. Frequency parameters ω_p for cones with SS3 ($\alpha = 60°$, $L/R_2 = 0.5$)

h/R_2	Without coupling ω_{pi}	ω_{pc}	ω_{pc}/ω_{pi}	With coupling ω_{pi}	ω_{pc}	ω_{pc}/ω_{pi}
0.01	0.2508	0.2522	1.005	0.2231	0.2328	1.043
0.02	0.3125	0.3162	1.012	0.2670	0.2856	1.070
0.03	0.3789	0.3889	1.026	0.3082	0.3354	1.088
0.04	0.4465	0.4687	1.050	0.3528	0.3891	1.103
0.05	0.5103	0.5511	1.080	0.3988	0.4462	1.119
0.06	0.5679	0.6325	1.114	0.4443	0.5053	1.137
0.07	0.6183	0.7102	1.149	0.4878	0.5650	1.158
0.08	0.6616	0.7815	1.181	0.5284	0.6244	1.182
0.09	0.6983	0.8444	1.209	0.5658	0.6825	1.206
0.10	0.7291	0.8973	1.231	0.5999	0.7383	1.231

the transverse shear deformation included in the improved shell theory has the effect of reducing the results predicted by the classical shell theory.

Tables 2–4 give the frequency parameters ω_{pc}, ω_{pi} and ω_{pc}/ω_{pi} for axisymmetric vibration of antisymmetric cross-plied laminated cones with the boundary condition of type 4, $L/R_2 = 0.5$ and different α, namely, Table 2 for $\alpha = 30°$, Table 3 for $\alpha = 45°$ and Table 4 for $\alpha = 60°$. In these Tables, it is noted that the results of the improved theory are lower than those of the classical theory, and the extension–bending coupling effects are the same for both theories in the sense of reducing the frequency parameters of the cones.

5. CONCLUSIONS

The salient points in this study include: (1) a systematic solution procedure is developed for free vibration analysis of laminated conical shells including the transverse shear deformation effect by using the power series method; (2) the solutions are applicable to all types of boundary conditions and to various kinds of isotropic, orthotropic and composite laminated conical shells; (3) by way of verification, the frequency parameters are computed for axisymmetric vibrations of cones with different geometric and material parameters and under various types of boundary conditions. Illustrative results show that: (a) the transverse shear deformation reduces the frequency parameters predicted by the classical theory; (b) the extension–bending coupling decreases the frequency parameters computed using either the improved or the classical theory.

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APPENDIX

The coefficients in eqns (12) are given by:

$$F_{1,1} = \kappa_3 G_{1,1} - \kappa_2 G_{4,1}, \quad F_{1,2} = \kappa_3 G_{1,2} - \kappa_2 G_{4,2}, \quad F_{1,3} = \kappa_3 G_{1,3},$$

$$F_{1,4} = \kappa_3 G_{1,4}, \quad F_{1,5} = \kappa_3 G_{1,5} - \kappa_2 G_{4,3}, \quad F_{1,6} = \kappa_3 G_{1,6} - \kappa_2 G_{4,4},$$

$$F_{1,7} = \kappa_3 G_{1,7} - \kappa_2 G_{4,5}, \quad F_{1,8} = \kappa_3 G_{1,8} - \kappa_2 G_{4,6}, \quad F_{1,9} = -\kappa_2 G_{4,7},$$

$$F_{1,10} = \kappa_3 G_{1,9} - \kappa_2 G_{4,8}, \quad F_{1,11} = \kappa_3 G_{1,10} - \kappa_2 G_{4,9}, \quad F_{1,12} = -\kappa_2 G_{4,10},$$

$$F_{1,13} = -\kappa_2 G_{4,11}, \quad F_{1,14} = \kappa_3 G_{1,11} - \kappa_2 G_{4,12}, \quad F_{1,15} = \kappa_3 G_{1,12} - \kappa_2 G_{4,13},$$

$$F_{2,1} = \lambda_3 G_{2,1} - \lambda_2 G_{5,1}, \quad F_{2,2} = \lambda_3 G_{2,2} - \lambda_2 G_{5,2}, \quad F_{2,3} = \lambda_3 G_{2,3} - \lambda_2 G_{5,3},$$

$$F_{2,4} = \lambda_3 G_{2,4} - \lambda_2 G_{5,4}, \quad F_{2,5} = \lambda_3 G_{2,5} - \lambda_2 G_{5,5}, \quad F_{2,6} = \lambda_3 G_{2,6},$$

$$F_{2,7} = \lambda_3 G_{2,7} - \lambda_2 G_{5,6}, \quad F_{2,8} = -\lambda_2 G_{5,7}, \quad F_{2,9} = \lambda_3 G_{2,8} - \lambda_2 G_{5,8},$$

$$F_{2,10} = \lambda_3 G_{2,9} - \lambda_2 G_{5,9}, \quad F_{2,11} = \lambda_3 G_{2,10} - \lambda_2 G_{5,10}, \quad F_{2,12} = \lambda_3 G_{2,11} - \lambda_2 G_{5,11},$$

$$F_{2,13} = \lambda_3 G_{2,12} - \lambda_2 G_{5,12}, \quad F_{2,14} = -\lambda_2 G_{5,13},$$

$$F_{3,i} = G_{3,i}/A_{55} \quad (i = 1, 2, 3, \ldots, 12)$$

$$F_{4,1} = -\kappa_2 G_{1,1} + \kappa_1 G_{4,1}, \quad F_{4,2} = -\kappa_2 G_{1,2} + \kappa_1 G_{4,2}, \quad F_{4,3} = -\kappa_2 G_{1,3},$$

$$F_{4,4} = -\kappa_2 G_{1,4}, \quad F_{4,5} = -\kappa_2 G_{1,5} + \kappa_1 G_{4,3}, \quad F_{4,6} = -\kappa_2 G_{1,6} + \kappa_1 G_{4,4},$$

$$F_{4,7} = -\kappa_2 G_{1,7} + \kappa_1 G_{4,5}, \quad F_{4,8} = -\kappa_2 G_{1,8} + \kappa_1 G_{4,6}, \quad F_{4,9} = \kappa_1 G_{4,7},$$

$$F_{4,10} = -\kappa_2 G_{1,9} + \kappa_1 G_{4,8}, \quad F_{4,11} = -\kappa_2 G_{1,10} + \kappa_1 G_{4,9}, \quad F_{4,12} = \kappa_1 G_{4,10},$$

$$F_{4,13} = \kappa_1 G_{4,11}, \quad F_{4,14} = -\kappa_2 G_{1,11} + \kappa_1 G_{4,12}, \quad F_{4,15} = -\kappa_2 G_{1,12} + \kappa_1 G_{4,13},$$

$$F_{5,1} = -\lambda_2 G_{2,1} + \lambda_1 G_{5,1}, \quad F_{5,2} = -\lambda_2 G_{2,2} + \lambda_1 G_{5,2}, \quad F_{5,3} = -\lambda_2 G_{2,3} + \lambda_1 G_{5,3},$$

$$F_{5,4} = -\lambda_2 G_{2,4} + \lambda_1 G_{5,4}, \quad F_{5,5} = -\lambda_2 G_{2,5} + \lambda_1 G_{5,5}, \quad F_{5,6} = -\lambda_2 G_{2,6},$$

$$F_{5,7} = -\lambda_2 G_{2,7} + \lambda_1 G_{5,6}, \quad F_{5,8} = \lambda_1 G_{5,7}, \quad F_{5,9} = -\lambda_2 G_{2,8} + \lambda_1 G_{5,8},$$

$$F_{5,10} = -\lambda_2 G_{2,9} + \lambda_1 G_{5,9}, \quad F_{5,11} = -\lambda_2 G_{2,10} + \lambda_1 G_{5,10}, \quad F_{5,12} = -\lambda_2 G_{2,11} + \lambda_1 G_{5,11},$$
$$F_{5,13} = -\lambda_2 G_{2,12} + \lambda_1 G_{5,12}, \quad F_{5,14} = \lambda_1 G_{5,13},$$

where

$$\kappa_0 = A_{11}D_{11} - B_{11}^2 \quad \kappa_1 = A_{11}/\kappa_0 \quad \kappa_2 = B_{11}/\kappa_0 \quad \kappa_3 = D_{11}/\kappa_0,$$
$$\lambda_0 = A_{33}D_{33} - B_{33}^2 \quad \lambda_1 = A_{33}/\lambda_0 \quad \lambda_2 = B_{33}/\lambda_0 \quad \lambda_3 = D_{33}/\lambda_0,$$

and the coefficients $G_{i,j}$ are:

$$G_{1,1} = -\frac{A_{11}(2m+1)\sin\alpha}{R_o(m+2)}, \quad G_{1,2} = -\frac{(A_{11}m^2 - A_{22})\sin^2\alpha - A_{33}n^2}{R_o^2(m+2)(m+1)} - \frac{\rho h\omega^2}{(m+2)(m+1)},$$

$$G_{1,3} = -\frac{2\rho h\omega^2 \sin\alpha}{R_o(m+2)(m+1)}, \quad G_{1,4} = -\frac{\rho h\omega^2 \sin^2\alpha}{R_o^2(m+2)(m+1)},$$

$$G_{1,5} = -\frac{A_{123}n}{R_o(m+2)}, \quad G_{1,6} = \frac{(-A_{123}m + A_{223})n\sin\alpha}{R_o^2(m+2)(m+1)},$$

$$G_{1,7} = -\frac{A_{12}\cos\alpha}{R_o(m+2)}, \quad G_{1,8} = \frac{(-A_{12}m + A_{22})\sin\alpha\cos\alpha}{R_o^2(m+2)(m+1)},$$

$$G_{1,9} = -\frac{B_{11}(2m+1)\sin\alpha}{R_o(m+2)}, \quad G_{1,10} = -\frac{(B_{11}m^2 - B_{22})\sin^2\alpha - B_{33}n^2}{R_o^2(m+2)(m+1)},$$

$$G_{1,11} = -\frac{B_{123}n}{R_o(m+2)}, \quad G_{1,12} = \frac{(-B_{123}m + B_{223})n\sin\alpha}{R_o^2(m+2)(m+1)},$$

$$G_{2,1} = \frac{A_{123}n}{R_o(m+2)}, \quad G_{2,2} = \frac{(A_{123}m + A_{223})n\sin\alpha}{R_o^2(m+2)(m+1)},$$

$$G_{2,3} = -\frac{A_{33}(2m+1)\sin\alpha}{R_o(m+2)}, \quad G_{2,4} = -\frac{A_{33}(m^2-1)\sin^2\alpha - A_{22}n^2 - A_{44}\cos^2\alpha}{R_o^2(m+2)(m+1)} - \frac{\rho h\omega^2}{(m+2)(m+1)}$$

$$G_{2,5} = -\frac{2\rho h\omega^2 \sin\alpha}{R_o(m+2)(m+1)}, \quad G_{2,6} = -\frac{\rho h\omega^2 \sin^2\alpha}{R_o^2(m+2)(m+1)},$$

$$G_{2,7} = \frac{(A_{22} + A_{44})n\cos\alpha}{R_o^2(m+2)(m+1)}, \quad G_{2,8} = \frac{B_{123}n}{R_o(m+2)},$$

$$G_{2,9} = \frac{(B_{123}m + B_{223})n\sin\alpha}{R_o^2(m+2)(m+1)}, \quad G_{2,10} = -\frac{B_{33}(2m+1)\sin\alpha}{R_o(m+2)},$$

$$G_{2,11} = -\frac{B_{33}(m^2-1)\sin^2\alpha - B_{22}n^2 + A_{44}R_o\cos\alpha}{R_o^2(m+2)(m+1)}, \quad G_{2,12} = -\frac{A_{44}\sin\alpha\cos\alpha}{R_o^2(m+2)(m+1)},$$

$$G_{3,1} = \frac{A_{12}\cos\alpha}{R_o(m+2)}, \quad G_{3,2} = \frac{(A_{12}m + A_{22})\sin\alpha\cos\alpha}{R_o^2(m+2)(m+1)},$$

$$G_{3,3} = \frac{(A_{22} + A_{44})n\cos\alpha}{R_o^2(m+2)(m+1)}, \quad G_{3,4} = -\frac{A_{55}(2m+1)\sin\alpha}{R_o(m+2)},$$

$$G_{3,5} = -\frac{A_{55}m^2\sin^2\alpha - A_{44}n^2 - A_{22}\cos^2\alpha}{R_o^2(m+2)(m+1)} - \frac{\rho h\omega^2}{(m+2)(m+1)},$$

$$G_{3,6} = -\frac{2\rho h\omega^2 \sin\alpha}{R_o(m+2)(m+1)}, \quad G_{3,7} = -\frac{\rho h\omega^2 \sin^2\alpha}{R_o^2(m+2)(m+1)},$$

$$G_{3,8} = \frac{B_{12}\cos\alpha - A_{55}R_o}{R_o(m+2)}, \quad G_{3,9} = -\frac{A_{55}R_o(2m+1)\sin\alpha - (B_{12}m + B_{22})\sin\alpha\cos\alpha,}{R_o^2(m+2)(m+1)},$$

$$G_{3,10} = -\frac{A_{55}m\sin^2\alpha}{R_o^2(m+2)(m+1)}, \quad G_{3,11} = \frac{(B_{22}\cos\alpha - A_{44}R_o)n}{R_o^2(m+2)(m+1)},$$

$$G_{3,12} = -\frac{A_{44}n\sin\alpha}{R_o^2(m+2)(m+1)},$$

$$G_{4,1} = -\frac{B_{11}(2m+1)\sin\alpha}{R_o(m+2)}, \quad G_{4,2} = -\frac{(B_{11}m^2 - B_{22})\sin^2\alpha - B_{33}n^2}{R_o^2(m+2)(m+1)},$$

$$G_{4,3} = -\frac{B_{123}n}{R_o(m+2)}, \quad G_{4,4} = \frac{(-B_{123}m + B_{223})n\sin\alpha}{R_o^2(m+2)(m+1)},$$

$$G_{4,5} = -\frac{B_{12}\cos\alpha - A_{55}R_o}{R_o(m+2)}, \quad G_{4,6} = -\frac{[(B_{12}m - B_{22})\cos\alpha - 2A_{55}R_o m]\sin\alpha}{R_o^2(m+2)(m+1)},$$

$$G_{4,7} = \frac{A_{55}(m-1)\sin^2\alpha,}{R_o^2(m+2)(m+1)}, \quad G_{4,8} = -\frac{D_{11}(2m+1)\sin\alpha}{R_o(m+2)},$$

$$G_{4,9} = -\frac{(D_{11}m^2 - D_{22})\sin^2\alpha - D_{33}n^2 - A_{55}R_o^2}{R_o^2(m+2)(m+1)} - \frac{\rho h^3\omega^2}{12(m+2)(m+1)},$$

$$G_{4,10} = \frac{2\left(A_{55} - \frac{\rho h^3\omega^2}{12}\right)\sin\alpha}{R_o(m+2)(m+1)}, \quad G_{4,11} = \frac{\left(A_{55} - \frac{\rho h^3\omega^2}{12}\right)\sin^2\alpha}{R_o^2(m+2)(m+1)},$$

$$G_{4,12} = -\frac{D_{123}n}{R_o(m+2)}, \quad G_{4,13} = \frac{(-D_{123}m + D_{223})n\sin\alpha,}{R_o^2(m+2)(m+1)},$$

$$G_{5,1} = \frac{B_{123}n}{R_o(m+2)}, \quad G_{5,2} = \frac{(B_{123}m + B_{223})n\sin\alpha}{R_o^2(m+2)(m+1)},$$

$$G_{5,3} = -\frac{B_{33}(2m+1)\sin\alpha}{R_o(m+2)}, \quad G_{5,4} = -\frac{B_{33}(m^2-1)\sin^2\alpha - \mathrm{B}_{22}n^2 + A_{44}R_o\cos\alpha}{R_o^2(m+2)(m+1)},$$

$$G_{5,5} = -\frac{A_{44}\sin\alpha\cos\alpha}{R_o^2(m+2)(m+1)}, \quad G_{5,6} = \frac{(B_{22}\cos\alpha - A_{44}R_o)n}{R_o^2(m+2)(m+1)},$$

$$G_{5,7} = -\frac{A_{44}n\sin\alpha}{R_o^2(m+2)(m+1)}, \quad G_{5,8} = \frac{D_{123}n}{R_o(m+2)},$$

$$G_{5,9} = \frac{(D_{123}m + D_{223})n\sin\alpha}{R_o^2(m+2)(m+1)}, \quad G_{5,10} = -\frac{D_{33}(2m+1)\sin\alpha}{R_o(m+2)},$$

$$G_{5,11} = -\frac{D_{33}(m^2-1)\sin^2\alpha - D_{22}n^2 - A_{44}R_o^2}{R_o^2(m+2)(m+1)} - \frac{\rho h^3\omega^2}{12(m+2)(m+1)},$$

$$G_{5,12} = \frac{2\left(A_{44} - \frac{\rho h^3\omega^2}{12}\right)\sin\alpha}{R_o(m+2)(m+1)}, \quad G_{5,13} = \frac{\left(A_{44} - \frac{\rho h^3\omega^2}{12}\right)\sin^2\alpha}{R_o^2(m+2)(m+1)},$$

where

$$A_{12} = A_{12} + A_{33} \quad A_{223} = A_{22} + A_{33}$$
$$B_{123} = B_{12} + B_{33} \quad B_{223} = B_{22} + B_{33}$$
$$D_{123} = D_{12} + D_{33} \quad D_{223} = D_{22} + D_{33}.$$